Stability of cnoidal waves for the focusing nonlinear Schrödinger equation with potential ⋆

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Abstract. We study the stability of the pulse solutions and the periodic solutions with large spatial periods of the focusing nonlinear Schrödinger equation with a potential. We play with symmetries of the equation and consider the edge bifurcation of the pulse solutions. Our results show spectral instability for all $V_0 > 0$, which is a free parameter in the potential, and spectral stability of the essential spectrum near the origin for $V_0 < 0$ close to zero. We also determined the instability mechanism for $V_0 > 0$.

1 Introduction

We consider the focusing cubic nonlinear Schrödinger equation (NLS) with potential studied in [2]

$$\mathrm{i}\psi_t = -\frac{1}{2}\psi_{xx} - |\psi|^2\psi + V(x)\psi \tag{1.1}$$

which models quasi-one-dimensional dilute-gas Bose-Einstein condensates (BEC) with attractive atomic interaction trapped in an external potential, for example standing light waves, and where $\psi(x,t)$ is the macroscopic wave function of the BEC and

$$V(x) = -V_0 \operatorname{sn}^2(x,k) \equiv V_0(\operatorname{cn}^2(x,k) - 1) \tag{1.2}$$

is a class of periodic potentials with the elliptic modulus $k \in [0,1]$.

For solutions of the form $\psi(x,t) = \phi(x,t)\mathrm{e}^{\mathrm{i}\omega t}$, the equation (1.1) becomes

$$\mathrm{i}\phi_t - \omega\phi = -\frac{1}{2}\phi_{xx} - |\phi|^2\phi + V(x)\phi. \tag{1.3}$$

We find the explicit formulas for $V_0 \geq -k^2$

$$\begin{aligned} \phi &= \frac{\sqrt{V_0+k^2}}{k}\operatorname{dn}(x,k)\ , \quad \omega = 1 + \frac{V_0}{k^2} - \frac{k^2}{2}\ , \quad \text{with period } 2L(k) \text{ or,} \\ \phi &= \sqrt{V_0+k^2}\operatorname{cn}(x,k)\ , \quad \omega = V_0 + k^2 - \frac{1}{2}\ , \quad \text{with period } 4L(k), \end{aligned} \tag{1.4}$$

where

$$L(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \to \infty \quad \text{as } \ k \to 1^-, \tag{1.5}$$

⋆ Preprint: Dec. 23, 2005

and $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ are the Jacobian elliptic cosine and delta amplitude functions, respectively, sometimes referred to as cnoidal waves. There are many other explicit solutions; see, for example, [2]. Numerically, they observed that the cn solutions are stable for $V_0 < 0$ and unstable for $V_0 > 0$ in [2]. However, the cnoidal cn waves fall into the regime where their instability criterion is not decisive since the cn solutions have zeros. Either way, the sign criterion they used is necessary, but never sufficient, for stability; it is really a parity instability criterion similar to the usual Evans function argument for an odd number of unstable real eigenvalues. That is the reason we study this problem; proving stability requires to actually compute the spectra of spatially periodic standing waves on the real line to account for modulational instabilities.

Remark 1.1. As $k \to 1$ **(hyperbolic limit)** the potential $V(x)$ becomes $-V_0 \tanh^2(x)$ and dn/cn waves approach a pulse, $\sqrt{V_0+1}\,\mathrm{sech}(x)$, with $\omega = V_0 + \frac{1}{2}$. As $V_0 \to 0$ **(pure NLS limit)** there exists a pulse solution, $\mathrm{sech}(x)$, with $\omega = \frac{1}{2}$ and dn/cn periodic waves accompany the pulse.

2 The nonlinear Schrödinger equation with potential

Linearizing (1.3) about a t-independent real-valued solution $q(x)$, then separating into real u and imaginary v parts give the eigenvalue problem

$$\mathcal{L}[q]\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.1}$$

where

$$\begin{aligned} L_+ &= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \omega + [V(x) - 3q^2(x)], \\ L_- &= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \omega + [V(x) - q^2(x)]. \end{aligned} \tag{2.2}$$

The equations $L_+ u = -\lambda v$, $L_- v = \lambda u$ can also be written as the first-order system

$$\begin{cases} u' = \tilde{u} \\ \tilde{u}' = 2(\omega + V - 3q^2)u + 2\lambda v \\ v' = \tilde{v} \\ \tilde{v}' = 2(\omega + V - q^2)v - 2\lambda u \end{cases} \qquad \text{i.e.,} \qquad W' = [A_q(x) + \lambda B]W \tag{2.3}$$

with

$$W = \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}, \quad A_q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2(\omega + V - 3q^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2(\omega + V - q^2) & 0 \end{pmatrix}. \tag{2.4}$$

We can easily check the following properties: recall that q is a real-valued function of x that satisfies $-\frac{1}{2}q'' + \omega q - q^3 + V(x)q = 0$. For $V_0 = 0$, $L_+ q' = 0$ and moreover $L_+(q + xq') = -q$ for $\omega = \frac{1}{2}$, i.e., when q is the sech pulse of potential-less NLS. For any V_0 and ω, $L_- q = 0$ and $L_-(xq) = -q'$. In (u, v) variables, these facts about $L_\pm$ translate to the following statements about the $\lambda = 0$ eigenvalue:

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0 & L_-\\ -L_+ & 0\end{pmatrix}\begin{pmatrix}q'\\0\end{pmatrix}, \quad \begin{pmatrix}q'\\0\end{pmatrix} \text{translation eigenfunction,}$$

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0 & L_-\\ -L_+ & 0\end{pmatrix}\begin{pmatrix}0\\q\end{pmatrix}, \quad \begin{pmatrix}0\\q\end{pmatrix} \text{rotation eigenfunction,}$$

$$\begin{pmatrix}q'\\0\end{pmatrix} = \begin{pmatrix}0 & L_-\\ -L_+ & 0\end{pmatrix}\begin{pmatrix}0\\-xq\end{pmatrix}, \quad \begin{pmatrix}0\\-xq\end{pmatrix} \text{generalized translation eigenfunction,}$$

$$\begin{pmatrix}0\\q\end{pmatrix} = \begin{pmatrix}0 & L_-\\ -L_+ & 0\end{pmatrix}\begin{pmatrix}q+xq'\\0\end{pmatrix}, \quad \begin{pmatrix}q+xq'\\0\end{pmatrix} \text{generalized rotation eigenfunction.}$$

Going to (u, u', v, v') variables, we have that

$$\Phi_T = \begin{pmatrix}q'\\q''\\0\\0\end{pmatrix}, \ \Phi_R = \begin{pmatrix}0\\0\\q\\q'\end{pmatrix} \text{ solve the variational equation } W' = A_q(x)W;$$

$$\Psi_T = \begin{pmatrix}-q''\\q'\\0\\0\end{pmatrix}, \ \Psi_R = \begin{pmatrix}0\\0\\-q'\\q\end{pmatrix} \text{ solve the adjoint variational equation } W' = -[A_q(x)]^* W; \text{ and}$$

$$\widehat{\Phi}_T = \begin{pmatrix}0\\0\\-xq\\-(xq)'\end{pmatrix}, \ \widehat{\Phi}_R = \begin{pmatrix}q+xq'\\(q+xq')'\\0\\0\end{pmatrix} \text{ solve } W' = A_q(x)W + B\Phi.$$

Note that if $q(x)$ is the sech pulse or one of the cnoidal waves shown in (1.4), then q and $q + xq'$ are even functions, while q' and $-xq$ are odd functions. Also, L_+ and L_- both map even functions to even functions, and odd functions to odd functions.

3 Spectra of pulses

We have mentioned that each of the operators L_+ and L_- maps odd functions to odd functions; moreover, L_- is invertible on the set of odd functions because $\ker(L_-) = \mathrm{Span}\{q\}$, and here we take $q(\cdot)$ to be even (it will be the sech pulse or the cn wave). Therefore, by using $(L_-)^{-1}$, the eigenvalue equations $L_+ u = -\lambda v$, $L_- v = \lambda u$ combine to give $L_- L_+ u = -\lambda^2 u$. Or, letting $\nu = \lambda^2$:

$$L_- L_+ u = -\nu u \tag{3.1}$$

When $V_0 = 0$, on the set of odd functions we have $\ker(L_- L_+) = \ker(L_+) = \mathrm{Span}\{q'\}$ from section 2, i.e., when $V_0 = 0$, (3.1) is satisfied by $\nu = 0$, $u = u_0 \equiv q'$.

3.1 Translation and rotation eigenvalues

Let us write $\epsilon \equiv V_0$, and expand in powers of ϵ. Then we have the following expansions:

$$\nu = 0 + b\epsilon + \mathcal{O}(\epsilon^2), \tag{3.2}$$

$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2) \quad \text{with } \langle u_0, u_1\rangle = 0, \tag{3.3}$$

$$L_- = L_-^0 + \epsilon L_-^1 + \mathcal{O}(\epsilon^2), \tag{3.4}$$

$$L_+ = L_+^0 + \epsilon L_+^1 + \mathcal{O}(\epsilon^2). \tag{3.5}$$

Note that $L_+^0 u_0 = 0$ and $L_+^1 = \frac{\partial}{\partial \epsilon} L_+\Big|_{\epsilon=0} = \left[\frac{\partial \omega}{\partial V_0} + \frac{\partial V(x)}{\partial V_0} - 6q\frac{\partial q}{\partial V_0}\right]_{V_0=0}$.

Substituting the expansions into (3.1) gives

$$L_-^0 L_+^0 u_0 + \epsilon\left(L_-^0 L_+^0 u_1 + L_-^0 L_+^1 u_0 + L_-^1 L_+^0 u_0\right) = -\epsilon b u_0 + \mathcal{O}(\epsilon^2). \tag{3.6}$$

Using $L_+^0 u_0 = 0$ to simplify, and collecting terms of $\mathcal{O}(\epsilon)$, we obtain

$$\begin{aligned} L_-^0(L_+^0 u_1 + L_+^1 u_0) &= -b u_0, \\ L_+^0 u_1 + L_+^1 u_0 &= -b(L_-^0)^{-1} u_0, \\ \langle L_+^0 u_1 + L_+^1 u_0, u_0\rangle &= -b\langle (L_-^0)^{-1} u_0, u_0\rangle, \\ \langle u_1, L_+^0 u_0\rangle + \langle L_+^1 u_0, u_0\rangle &= -b\langle (L_-^0)^{-1} u_0, u_0\rangle. \end{aligned} \tag{3.7}$$

Thus, we have

$$b = -\frac{\langle L_+^1 u_0, u_0\rangle}{\langle (L_-^0)^{-1} u_0, u_0\rangle}. \tag{3.8}$$

Further note that as $L_-(xq) = -q'$, we have $(L_-^0)^{-1} u_0 = (L_-^0)^{-1} q' = -xq$ and hence the denominator becomes $\langle (L_-^0)^{-1} u_0, u_0\rangle = \int -xqq' = \frac{1}{2}\int q^2$ after integration by parts.

For the pulse at $k = 1$:

Since $V(x) = -V_0 \tanh^2(x)$, $q = \sqrt{V_0+1}\,\mathrm{sech}(x)$, and $\omega = V_0 + \frac{1}{2}$, we have $\frac{\partial V(x)}{\partial V_0} = -\tanh^2(x)$, $6q\frac{\partial q}{\partial V_0} = 3\,\mathrm{sech}^2(x)$, and $\frac{\partial \omega}{\partial V_0} = 1$. So we obtain

$$L_+^1 u_0 = [1 - \tanh^2(x) - 3\,\mathrm{sech}^2(x)]q' = -2\,\mathrm{sech}^2(x) q', \tag{3.9}$$

$$\begin{aligned} \langle L_+^1 u_0, u_0\rangle &= -2\int_{-\infty}^{\infty} \mathrm{sech}^2(x)\cdot (q')^2\, dx \\ &= -2(V_0+1)\int_{-\infty}^{\infty} \mathrm{sech}^2(x)\cdot\, \tanh^2(x)\,\mathrm{sech}^2(x)\, dx, \end{aligned} \tag{3.10}$$

$$\langle (L_-^0)^{-1} u_0, u_0\rangle = \frac{1}{2}(V_0+1)\int_{-\infty}^{\infty} \mathrm{sech}^2(x)\, dx, \tag{3.11}$$

hence

$$b = 4\,\frac{\int_{-\infty}^{\infty} \mathrm{sech}^4(x)\tanh^2(x)\, dx}{\int_{-\infty}^{\infty} \mathrm{sech}^2(x)\, dx} = \frac{8}{15}. \tag{3.12}$$

For the cn waves at $k < 1$:

Since $V(x) = V_0(\mathrm{cn}^2(x,k) - 1)$, $q = \sqrt{V_0 + k^2}\,\mathrm{cn}(x,k)$, and $\omega = V_0 + k^2 - \frac{1}{2}$, we have $\dfrac{\partial V(x)}{\partial V_0} = \mathrm{cn}^2(x,k) - 1$, $6q\dfrac{\partial q}{\partial V_0} = 3\,\mathrm{cn}^2(x,k)$, and $\dfrac{\partial \omega}{\partial V_0} = 1$. So we obtain

$$L_+^1 u_0 = [1 + \mathrm{cn}^2(x,k) - 1 - 3\,\mathrm{cn}^2(x,k)]q' = -2\,\mathrm{cn}^2(x,k)q', \tag{3.13}$$

$$\begin{aligned}\langle L_+^1 u_0, u_0\rangle &= -2\int_{-2L(k)}^{2L(k)} \mathrm{cn}^2(x,k)\cdot(q')^2\,dx\\ &= -2(V_0 + k^2)\int_{-2L(k)}^{2L(k)} \mathrm{cn}^2(x,k)\cdot\,\mathrm{dn}^2(x,k)\,\mathrm{sn}^2(x,k)\,dx,\end{aligned} \tag{3.14}$$

$$\langle (L_-^0)^{-1}u_0, u_0\rangle = \frac{1}{2}(V_0 + k^2)\int_{-2L(k)}^{2L(k)} \mathrm{cn}^2(x,k)\,dx, \tag{3.15}$$

hence

$$b = 4\,\frac{\int_{-2L(k)}^{2L(k)} \mathrm{cn}^2(x,k)\,\mathrm{dn}^2(x,k)\,\mathrm{sn}^2(x,k)\,dx}{\int_{-2L(k)}^{2L(k)} \mathrm{cn}^2(x,k)\,dx} > 0, \quad \forall k \in (0,1). \tag{3.16}$$

For the dn waves at $k < 1$:

Since $V(x) = -V_0\,\mathrm{sn}^2(x,k)$, $q = \dfrac{\sqrt{V_0 + k^2}}{k}\,\mathrm{dn}(x,k)$, and $\omega = 1 + \dfrac{V_0}{k^2} - \dfrac{k^2}{2}$, we have $\dfrac{\partial V(x)}{\partial V_0} = -\mathrm{sn}^2(x,k)$, $6q\dfrac{\partial q}{\partial V_0} = \dfrac{3}{k^2}\,\mathrm{dn}^2(x,k)$, and $\dfrac{\partial \omega}{\partial V_0} = \dfrac{1}{k^2}$. So we obtain

$$\begin{aligned}L_+^1 u_0 &= \left[\frac{1}{k^2} - \mathrm{sn}^2(x,k) - \frac{3}{k^2}\,\mathrm{dn}^2(x,k)\right] q'\\ &= \frac{1}{k^2}[1 - k^2\,\mathrm{sn}^2(x,k) - 3\,\mathrm{dn}^2(x,k)]\,q'\\ &= \frac{1}{k^2}[\,\mathrm{dn}^2(x,k) - 3\,\mathrm{dn}^2(x,k)]\,q' = \frac{-2}{k^2}\,\mathrm{dn}^2(x,k)\,q',\end{aligned} \tag{3.17}$$

$$\begin{aligned}\langle L_+^1 u_0, u_0\rangle &= \frac{-2}{k^2}\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\cdot(q')^2\,dx\\ &= \frac{-2}{k^2}\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\cdot\left(\frac{\sqrt{V_0+k^2}}{k}\cdot(-k^2)\,\mathrm{sn}(x,k)\,\mathrm{cn}(x,k)\right)^2 dx\\ &= -2(V_0 + k^2)\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\cdot\,\mathrm{sn}^2(x,k)\,\mathrm{cn}^2(x,k)\,dx,\end{aligned} \tag{3.18}$$

$$\langle (L_-^0)^{-1}u_0, u_0\rangle = \frac{(V_0 + k^2)}{2k^2}\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\,dx, \tag{3.19}$$

hence

$$b = 4k^2\,\frac{\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\,\mathrm{sn}^2(x,k)\,\mathrm{cn}^2(x,k)\,dx}{\int_{-L(k)}^{L(k)} \mathrm{dn}^2(x,k)\,dx} > 0, \quad \forall k \in (0,1). \tag{3.20}$$

In summary, for $V_0 \neq 0$ and $k \in (0,1]$, we have $\nu = bV_0 + \mathcal{O}(V_0^2)$, i.e.,

$$\begin{aligned}\lambda^2 &= bV_0 + \mathcal{O}(V_0^2),\\ \lambda &= \pm\sqrt{V_0(b+\mathcal{O}(V_0))}\end{aligned} \tag{3.21}$$

where $b > 0$ and $b = \frac{8}{15}$ for $k = 1$. This implies that for all $k \in (0,1]$, to leading order the eigenvalues are imaginary if $V_0 < 0$ (linear stability), whereas in the case of $V_0 > 0$ there is spectrum off the imaginary axis (linear instability).

3.2 Edge bifurcations

Consider the pulse $q = \operatorname{sech}(x)$ at $V_0 = 0$, with $\omega = \frac{1}{2}$. The essential spectrum associated with this solution of the partial differential equation (PDE) is $\sigma_{\text{ess}} = \{\lambda \in \mathrm{i}\mathbb{R} \ : \ |\lambda| \geq \frac{1}{2}\}$. The eigenvalue problem can be expressed as

$$\left[\left(\frac{\partial^2}{\partial x^2} - 1\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 2q^2\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix} + 2\lambda\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}u\\ v\end{pmatrix} = 0. \tag{3.22}$$

Parametrising by $\lambda = \frac{\mathrm{i}}{2}(1-\mu^2)$ indicates that the function

$$\mathcal{V}^{\mathrm{u}} = \mathrm{e}^{\mu x}\left[-\frac{1}{2}(1+\mu^2 - 2\mu\tanh(x))\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} + \operatorname{sech}^2(x)\begin{pmatrix}1\\ 0\end{pmatrix}\right] \tag{3.23}$$

solves $\mathcal{L}[q]\begin{pmatrix}u\\ v\end{pmatrix} = \frac{\mathrm{i}}{2}(1-\mu^2)\begin{pmatrix}u\\ v\end{pmatrix}$ for $\mu > 0$ when $V_0 = 0$.

As a first-order system, the eigenvalue problem (2.3) is

$$\begin{aligned}&u' = \tilde{u}, \qquad v' = \tilde{v},\\ &\tilde{u}' = [1 - 6\operatorname{sech}^2(x) - 4V_0\operatorname{sech}^2(x)]u + \mathrm{i}(1-\mu^2)v,\\ &\tilde{v}' = [1 - 2\operatorname{sech}^2(x)]v - \mathrm{i}(1-\mu^2)u,\end{aligned} \tag{3.24}$$

upon taking $\omega = V_0 + \frac{1}{2}$, $V = -V_0\tanh^2(x)$, $q = \sqrt{V_0+1}\operatorname{sech}(x)$. We rewrite it as

$$Y' = \begin{pmatrix}0 & I\\ M & 0\end{pmatrix}Y \quad \text{where } Y = \begin{pmatrix}u\\ v\\ u'\\ v'\end{pmatrix}, \quad M = \begin{pmatrix}1 - 6\operatorname{sech}^2(x) - 4V_0\operatorname{sech}^2(x) & \mathrm{i}(1-\mu^2)\\ -\mathrm{i}(1-\mu^2) & 1 - 2\operatorname{sech}^2(x)\end{pmatrix}. \tag{3.25}$$

Then $\mathcal{Y}^{\mathrm{u}} = \begin{pmatrix}\mathcal{V}^{\mathrm{u}}\\ (\mathcal{V}^{\mathrm{u}})'\end{pmatrix}$ is a solution when $V_0 = 0$, and its associated adjoint vector is $\mathcal{X} = \begin{pmatrix}-(\mathcal{V}^{\mathrm{u}})'\\ \mathcal{V}^{\mathrm{u}}\end{pmatrix}$.

What happens when we turn on V_0:

We extend the Evans function across the essential spectrum, using $\mathcal{V}^{\mathrm{u}}$ [4]; then we seek zeros of $E(\mu, V_0)$. We have the following relation:

$$\begin{aligned} \mu \in \mathrm{i}\mathbb{R} \quad &\leftrightarrow \quad \lambda \in \sigma_{\mathrm{ess}} \\ \mu \in \mathbb{R}^- \quad &\leftrightarrow \quad \text{resonance pole ('eigenfunction' blows up at } \pm\infty) \\ \mu \in \mathbb{R}^+ \quad &\leftrightarrow \quad \text{edge bifurcation} \end{aligned}$$

Using the notation introduced above, the extended Evans function is

$$\begin{aligned} E(\mu, V_0) &= \left\langle \mathcal{V}^{\mathrm{u}}, \frac{\partial M}{\partial V_0} \mathcal{V}^{\mathrm{u}} \right\rangle_{L^2} \bigg|_{\mu=0} V_0 + \left\langle \mathcal{X}^*, 2\frac{\partial \mathcal{V}^{\mathrm{u}}}{\partial \mu} \right\rangle \bigg|_{x=0,\mu=0} \mu + \text{ h.o.t.} \\ &= -\frac{14}{15} V_0 + 1 \cdot \mu + \text{ h.o.t.} \end{aligned} \tag{3.26}$$

To leading order, $E(\mu, V_0) = 0$ for $\mu = \frac{14}{15} V_0$ gives the edge bifurcation for $V_0 > 0$ or no edge bifurcation for $V_0 < 0$, and

$$\lambda_{\mathrm{edge}} = \frac{\mathrm{i}}{2} \left[1 - \left(\frac{14}{15} \right)^2 V_0^2 \right]. \tag{3.27}$$

4 Spectra of wave trains with large spatial period

Recall, from section 2, the eigenvalue problem $\mathcal{L}[q] \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$ written as a first-order system (2.3):

$$W' = [A_q(x) + \lambda B] W$$

with W, A_q, B defined as in (2.4). Here we will let q represent either a homoclinic solution $h(x)$ (sech pulse) or a periodic solution $p_L(x)$ (cnoidal wave).

In this section we shall determine the spectra of cnoidal wave-trains which approach the primary pulse with ever increasing period as the parameter k tends to 1. Gardner [3] showed that a long-wavelength periodic solution close to a homoclinic orbit possesses a loop of (essential) spectra in a neighborhood of each isolated eigenvalue of the primary pulse. By Floquet theory, the spectral stability problem for a $2L$-periodic wave $p_L(x)$ can be stated as follows:

$\lambda \in \mathbb{C}$ is in the spectrum of the linearisation $\mathcal{L}[p_L]$ if and only if there is a $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and a bounded function $W(x)$ such that

$$\begin{aligned} &W' = [A_{p_L}(x) + \lambda B] W \qquad \text{for } |x| < L, \\ &W(L) = \mathrm{e}^{\mathrm{i}\gamma} W(-L). \end{aligned} \tag{4.1}$$

If the primary pulse has eigenvalue(s) at 0 (as is the case here due to translation and rotation invariance), then for the $2L$-periodic wave trains that accompany the pulse, it is critical to determine the location of the loop of spectra near 0, i.e., to solve the boundary value problem (4.1) for λ close to 0: $\lambda(\gamma)$ will be parametrised by $\gamma \in [0, 2\pi)$ such that $\lambda(0) = 0$.

4.1 Review of Liapunov-Schmidt method

The general theory developed in [5] applies to our problem, subject to a few modifications that will be described later. The basic theorem that we apply (Theorem 2.2 of [5]) can be summarised as follows: under certain assumptions on the pulse h and accompanying periodic solution p_L, (4.1) has a solution for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and $\lambda \in \mathbb{C}$ close to 0 if and only if

$$\det E(\lambda, \gamma) = 0 \tag{4.2}$$

where the $m \times m$ matrix E is analytic in (λ, γ) and, for $1 \leq j, k \leq m$,

$$\begin{aligned} E_{kj}(\lambda, \gamma) = & (\mathrm{e}^{\mathrm{i}\gamma} - 1)\langle \Psi_k(L), \Phi_j(-L)\rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\langle \Psi_k(-L), \Phi_j(L)\rangle \\ & -\lambda \int_{-\infty}^{\infty} \langle \Psi_k(x), B\Phi_j(x)\rangle \, dx \\ & +(\mathrm{e}^{\mathrm{i}\gamma} - 1)R_{kj}(\lambda, \gamma) + \lambda \tilde{R}_{kj}(\lambda, \gamma) \end{aligned} \tag{4.3}$$

Here the Φ_j are linearly independent, bounded solutions to the variational equation $W' = A_h(x)W$ about the pulse, and the Ψ_k are linearly independent, bounded solutions to the adjoint variational equation $W' = -[A_h(x)]^*W$. Estimates for the remainder terms $R_{kj}(\lambda, \gamma)$, $\tilde{R}_{kj}(\lambda, \gamma)$ and their derivatives with respect to λ, γ are given in [5].

The assumptions are essentially that the periodic solution $p_L(\cdot)$ has sufficiently large period and is close enough to the homoclinic $h(\cdot)$; also h is assumed to be nondegenerate in the sense that solutions generated by symmetries of the PDE, i.e., the Φ_j, form a basis for all bounded solutions to the variational equation $W' = A_h(x)W$.

In our case, translation and rotation symmetry of the PDE with $V_0 = 0$ generate a two-dimensional eigenspace for $\lambda = 0$, i.e. the geometric multiplicity of $\lambda = 0$ is 2, and the set of all bounded solutions of $W' = A_h(x)W$ is spanned by two linearly independent functions $\Phi_T(x)$ and $\Phi_R(x)$. In other words, we have $m = 2$, and shall take $\Phi_1 = \Phi_T$, $\Phi_2 = \Phi_R$ and $\Psi_1 = \Psi_T$, $\Psi_2 = \Psi_R$, where all these functions are as described in section 2.

To understand what modifications should be made to this theorem in order to solve the problem at hand, let us review some main points in the derivation of $E(\lambda, \gamma)$. The derivation hinges on the Liapunov-Schmidt reduction method which replaces an equation $\mathcal{L}u = 0$, where $\mathcal{L} : X \to Y$ is a Fredholm operator, by a simpler equation of the form

$$\tilde{\mathcal{L}}u = 0, \qquad \tilde{\mathcal{L}} : N(\mathcal{L}) \to R(\mathcal{L})^{\perp}.$$

This reduces the dimension of the problem, as $\dim(R(\tilde{\mathcal{L}})) = \dim(Y) - \dim(R(\mathcal{L}))$.

First, (4.1) is rewritten in the extended form

$$\begin{aligned} W_-' &= A_h(x)W_- + [A_{p_L}(x) - A_h(x) + \lambda B]W_-, \quad x \in (-L, 0), \\ W_+' &= A_h(x)W_+ + [A_{p_L}(x) - A_h(x) + \lambda B]W_+, \quad x \in (0, L), \\ W_+(L) &= \mathrm{e}^{\mathrm{i}\gamma}W_-(-L), \end{aligned} \tag{4.4}$$

together with the matching condition $W_-(0) = W_+(0)$. Instead of requiring that $W_-(0) = W_+(0)$ be satisfied at the outset, system (4.4) is solved first, with a discontinuity $\xi(\lambda, \gamma)$ allowed at $x = 0$ within a two-dimensional subspace of $\mathbb{R}^4$ (because $m = 2$). Then, at the final step, we will solve $\xi = 0$, i.e. seek (λ, γ) such that W_+ and W_- are the same at $x = 0$.

To solve (4.4), write $W_\pm(x) = \sum_{j=1}^m d_j \phi_j(x) + w^\pm(x)$, where $\phi_j(x)$ satisfy $W' = A_{p_L}(x) W$ (the variational equation about the periodic solution) together with $W(L) = W(-L)$, and reformulate the problem in terms of $w^\pm$. Expressions for the jumps $w^+(0) - w^-(0)$ along the directions Ψ_k $(k = T, R)$, denoted $\xi_k \equiv \langle \Psi_k(0), w^+(0) - w^-(0) \rangle$, are then derived using variation-of-constants formulas, exponential dichotomies and the Liapunov-Schmidt reduction procedure. Proximity of the periodic solution to the homoclinic solution is exploited to derive error estimates that justify the formulas for the jumps.

We remark that in (4.3), the integral term arises from the inhomogeneous terms of the differential equations for $w^\pm$, upon applying the variation-of-constants formula. It is essentially a Melnikov integral which measures the jump size for small values of $\lambda \neq 0$, assuming $L = \infty$ (i.e. it measures how the $\lambda = 0$ eigenfunctions split as λ is turned on, while ignoring the boundary conditions at $\pm L$). This integral also has an interpretation as the derivative of the Evans function associated with the primary pulse h, so it vanishes if $\lambda = 0$ is an eigenvalue of algebraic multiplicity greater than 1. The inner product terms in (4.3), on the other hand, contain information about jump sizes when λ is taken to be 0, but with $L < \infty$. In other words, the effect of the boundary (Bloch-wave) condition $W_+(L) = \mathrm{e}^{\mathrm{i}\gamma} W_-(-L)$ is measured by these inner product terms.

Formula (4.3) cannot be applied directly to locate the spectra of the cnoidal waves. One reason is that the Melnikov integral terms

$$\lambda \int_{-\infty}^{\infty} \langle \Psi_k(x), B\Phi_j(x) \rangle \, dx \tag{4.5}$$

(where $k, j = T$ or R) are zero to $\mathcal{O}(\lambda)$, because the algebraic multiplicity is 2 for each eigenspace corresponding to the translation and rotation symmetries. To obtain the integral terms of next order, we need to compute the $\mathcal{O}(\lambda^2)$ jump, which is

$$\frac{1}{2}\lambda^2 \left.\frac{\partial^2 \xi}{\partial \lambda^2}\right|_{\lambda=0} = \frac{1}{2}\lambda^2 \left.\langle \Psi(0), w^+_{\lambda\lambda}(0) - w^-_{\lambda\lambda}(0) \rangle\right|_{\lambda=0} \equiv \frac{1}{2}\lambda^2 \langle \Psi(0), w^+_2(0) - w^-_2(0) \rangle \tag{4.6}$$

Differentiating $W' = [A(x) + \lambda B]W$ with respect to λ and setting $\lambda = 0$ gives

$$\begin{aligned} W' &= [A(x) + \lambda B]W, & w_0' &= A(x) w_0, \\ W_\lambda' &= [A(x) + \lambda B]W_\lambda + BW, & w_1' &= A(x) w_1 + B w_0, \\ W_{\lambda\lambda}' &= [A(x) + \lambda B]W_{\lambda\lambda} + 2BW_\lambda, & w_2' &= A(x) w_2 + 2B w_1. \end{aligned} \tag{4.7}$$

Applying the variation-of-constants formula to $w_2' = A(x)w_2 + 2Bw_1$ generates integrals of the form

$$\begin{aligned} &\frac{1}{2}\lambda^2 \left\langle \Psi(0), \int_{\infty}^{0} \Phi_u^+ \, 2Bw_1(x) dx - \int_{-\infty}^{0} \Phi_s^- \, 2Bw_1(x) dx \right\rangle \\ &= \lambda^2 \int_{-\infty}^{\infty} \langle \Psi(x), Bw_1(x) \rangle \, dx \end{aligned} \tag{4.8}$$

where Φ_u^+, Φ_s^- are evolution operators on $\mathbb{R}^+, \mathbb{R}^-$ respectively (see [5] for details). Hence

$$\lambda^2 \int_{-\infty}^{\infty} \langle \Psi(x), B\widehat{\Phi}(x) \rangle \, dx \tag{4.9}$$

is the next order integral term, where

$$\widehat{\Phi} \text{ solves } W' = A_h(x)W + B\Phi \tag{4.10}$$

$$\text{and} \quad \Phi \text{ solves } W' = A_h(x)W. \tag{4.11}$$

For the cn waves, there is an added complication that the inner product terms in (4.3) also need to be modified. This is because the formula was derived in [5] for a dn-type periodic solution which approaches one copy of the primary homoclinic solution as the period tends to infinity. The cn waves, in contrast, approach *two* symmetric copies of the sech pulse as $k \to 1$. With $W_\pm(x) = \sum d_j \phi_j(x) + w^\pm(x)$, where $j = T$ or R, the boundary condition $W_+(L) = \mathrm{e}^{\mathrm{i}\gamma} W_-(-L)$ now translates to

$$w^+(L) - \mathrm{e}^{\mathrm{i}\gamma} w^-(-L) = (1 + \mathrm{e}^{\mathrm{i}\gamma}) \sum d_j \phi_j(-L) = -(1 + \mathrm{e}^{\mathrm{i}\gamma}) \sum d_j \phi_j(L) \tag{4.12}$$

$$\text{instead of} \quad (\mathrm{e}^{\mathrm{i}\gamma} - 1) \sum d_j \phi_j(-L) = (\mathrm{e}^{\mathrm{i}\gamma} - 1) \sum d_j \phi_j(L) \tag{4.13}$$

because cn waves have $\phi(L) = -\phi(-L)$ instead of $\phi(L) = \phi(-L)$. Thus, the correct inner product terms for the cn waves are

$$\begin{aligned} &(1 + \mathrm{e}^{\mathrm{i}\gamma}) \langle \Psi(L), \Phi(-L) \rangle + \mathrm{e}^{-\mathrm{i}\gamma} [-(1 + \mathrm{e}^{\mathrm{i}\gamma})] \langle \Psi(-L), \Phi(L) \rangle \\ &= (1 + \mathrm{e}^{\mathrm{i}\gamma}) \langle \Psi(L), \Phi(-L) \rangle - (1 + \mathrm{e}^{-\mathrm{i}\gamma}) \langle \Psi(-L), \Phi(L) \rangle \end{aligned} \tag{4.14}$$

instead of

$$\begin{aligned} &(\mathrm{e}^{\mathrm{i}\gamma} - 1) \langle \Psi(L), \Phi(-L) \rangle + \mathrm{e}^{-\mathrm{i}\gamma} (\mathrm{e}^{\mathrm{i}\gamma} - 1) \langle \Psi(-L), \Phi(L) \rangle \\ = \quad &(\mathrm{e}^{\mathrm{i}\gamma} - 1) \langle \Psi(L), \Phi(-L) \rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma}) \langle \Psi(-L), \Phi(L) \rangle \end{aligned} \tag{4.15}$$

in the case of dn waves.

Therefore, for the cn waves, the corresponding formulas for the jumps at $x = 0$ are

$$\begin{aligned} \xi_{kj} = &(1 + \mathrm{e}^{\mathrm{i}\gamma}) \langle \Psi_k(L), \Phi_j(-L) \rangle - (1 + \mathrm{e}^{-\mathrm{i}\gamma}) \langle \Psi_k(-L), \Phi_j(L) \rangle \\ &- \lambda^2 \int_{-\infty}^{\infty} \langle \Psi_k(x), B\widehat{\Phi}_j(x) \rangle \, dx + \mathcal{R}_{kj}(\lambda, \gamma). \end{aligned} \tag{4.16}$$

4.2 Coefficients for translation eigenvalues

For $V_0 = 0$ and $k < 1$ but close to 1 (so that $L \gg 1$), we calculate the translation-related jump as follows. Taking $k = T, j = T$ in (4.16), and using the expressions

$$\Phi_T = \begin{pmatrix} q' \\ q'' \\ 0 \\ 0 \end{pmatrix}, \ \Psi_T = \begin{pmatrix} -q'' \\ q' \\ 0 \\ 0 \end{pmatrix}, \ B\widehat{\Phi}_T = \begin{pmatrix} 0 \\ -2xq \\ 0 \\ 0 \end{pmatrix}$$

with $q(x) = \mathrm{sech}(x)$, the homoclinic solution at $k = 1$ and $V_0 = 0$, we obtain

$$\begin{aligned}
\xi_{TT} &= (1 + \mathrm{e}^{\mathrm{i}\gamma})\langle \Psi_T(L), \Phi_T(-L)\rangle - (1 + \mathrm{e}^{-\mathrm{i}\gamma})\langle \Psi_T(-L), \Phi_T(L)\rangle \\
&\quad -\lambda^2 \int_{-\infty}^{\infty} \langle \Psi_T(x), B\widehat{\Phi}_T(x)\rangle \, dx + \mathcal{R}_{kj}(\lambda, \gamma) \\
&= (1 + \mathrm{e}^{\mathrm{i}\gamma}) \left[-q''(L)q'(-L) + q'(L)q''(-L)\right] \\
&\quad -(1 + \mathrm{e}^{-\mathrm{i}\gamma}) \left[-q''(-L)q'(L) + q'(-L)q''(L)\right] \\
&\quad +2\lambda^2 \int_{-\infty}^{\infty} q' \cdot xq \, dx + \mathcal{R}_{kj}(\lambda, \gamma) \\
&= 2(1 + \mathrm{e}^{\mathrm{i}\gamma})q'(L)q''(L) + 2(1 + \mathrm{e}^{-\mathrm{i}\gamma})q'(L)q''(L) \\
&\quad +2\lambda^2 \int_{-\infty}^{\infty} xqq' \, dx + \mathcal{R}_{kj}(\lambda, \gamma) \\
&= 4(1 + \cos\gamma)q'(L)q''(L) - \lambda^2 \int_{-\infty}^{\infty} q^2 \, dx + \mathcal{R}_{kj}(\lambda, \gamma)
\end{aligned} \tag{4.17}$$

using the reversible symmetry $q'(-x) = -q'(x), q''(-x) = q''(x)$. Since $q'(L)q''(L) \sim -4\mathrm{e}^{-2L}$ and $\int_{-\infty}^{\infty} q^2 \, dx = 2$ we have, to leading order

$$\xi_{TT} \approx -16(1 + \cos\gamma)\mathrm{e}^{-2L} - 2\lambda^2 \tag{4.18}$$

Thus, solving $\xi_{TT} = 0$ gives

$$\lambda^2 \approx -8(1 + \cos\gamma)\mathrm{e}^{-2L}. \tag{4.19}$$

This accords with the results in [1], obtained via a formal asymptotic analysis.

Note that, when $V_0 = 0$, we have $\lambda \approx \pm\mathrm{i}\sqrt{8}\sqrt{(1 + \cos\gamma)}\,\mathrm{e}^{-L}$. Since $(1 + \cos\gamma) \geq 0$, this means that, to leading order, the spectrum for the wavetrain arising from the translation eigenvalue of the pulse turns out to be purely imaginary, and hence does not necessarily generate instability. However, when $V_0 \neq 0$, the potential breaks translation invariance of the PDE; the pulse eigenvalue at 0 splits into two eigenvalues on the real axis, given by

$$\lambda^2 = \frac{8}{15}V_0 + \mathcal{O}(V_0^2) \tag{4.20}$$

as seen in section 3.

Combining (4.19) and (4.20) gives us the approximate position of the loops of spectra for the wavetrains when $V_0 \neq 0$:

$$\begin{aligned}
\lambda^2 &= \frac{8}{15}V_0 - 8(1 + \cos\gamma)\mathrm{e}^{-2L} \\
\lambda &= \pm\sqrt{8\left(\frac{1}{15}V_0 - (1 + \cos\gamma)\mathrm{e}^{-2L}\right)}
\end{aligned} \tag{4.21}$$

4.3 Coefficients for rotation eigenvalues

The rotation-related jump, for $V_0 = 0$ and $k < 1$ but close to 1, is calculated as follows. Taking $k = R, j = R$ in (4.16), and using the expressions

$$\Phi_R = \begin{pmatrix} 0 \\ 0 \\ q \\ q' \end{pmatrix}, \ \Psi_R = \begin{pmatrix} 0 \\ 0 \\ -q' \\ q \end{pmatrix}, \ B\widehat{\Phi}_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2(q + xq') \end{pmatrix}$$

with $q(x) = \operatorname{sech}(x)$, we obtain

$$\begin{aligned}
\xi_{RR} &= (1 + \mathrm{e}^{\mathrm{i}\gamma})\langle \Psi_R(L), \Phi_R(-L)\rangle - (1 + \mathrm{e}^{-\mathrm{i}\gamma})\langle \Psi_R(-L), \Phi_R(L)\rangle \\
&\quad -\lambda^2 \int_{-\infty}^{\infty} \langle \Psi_R(x), B\widehat{\Phi}_R(x)\rangle \, dx \\
&= (1 + \mathrm{e}^{\mathrm{i}\gamma}) \left[-q'(L)q(-L) + q(L)q'(-L)\right] \\
&\quad -(1 + \mathrm{e}^{-\mathrm{i}\gamma}) \left[-q'(-L)q(L) + q(-L)q'(L)\right] \\
&\quad +2\lambda^2 \int_{-\infty}^{\infty} q \cdot (q + xq') \, dx \\
&= -2(1 + \mathrm{e}^{\mathrm{i}\gamma})q(L)q'(L) - 2(1 + \mathrm{e}^{-\mathrm{i}\gamma})q(L)q'(L) \\
&\quad +2\lambda^2 \int_{-\infty}^{\infty} q^2 + xqq' \, dx \\
&= -4(1 + \cos\gamma)q(L)q'(L) + 2\lambda^2 \int_{-\infty}^{\infty} q^2 - \frac{1}{2}q^2 \, dx
\end{aligned} \tag{4.22}$$

Since $q(L)q'(L) \sim -4\mathrm{e}^{-2L}$ and $\int_{-\infty}^{\infty} q^2 - \frac{1}{2}q^2 \, dx = 1$ we have, to leading order

$$\xi_{RR} \approx 16(1 + \cos\gamma)\mathrm{e}^{-2L} + 2\lambda^2 \tag{4.23}$$

Thus, solving $\xi_{RR} = 0$ gives

$$\begin{aligned}
\lambda^2 &= -8(1 + \cos\gamma)\mathrm{e}^{-2L} \\
\lambda &= \pm\mathrm{i}\sqrt{8(1 + \cos\gamma)} \, \mathrm{e}^{-L}
\end{aligned} \tag{4.24}$$

to leading order. Note that addition of the potential term does not break rotation invariance; with $V_0 \neq 0$, q is still an eigenfunction of L_- for $\lambda = 0$ (see section 2). Therefore the above formula gives the position of the spectral loops for $V_0 \neq 0$ also.

Remark 4.1. In a similar fashion, the jumps ξ_{TT}, ξ_{RR} can also be calculated for the dn waves, using the unmodified inner product terms (4.15). That is,

$$\begin{aligned}
\xi_{TT} &= (\mathrm{e}^{\mathrm{i}\gamma} - 1)\langle \Psi_T(L), \Phi_T(-L)\rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\langle \Psi_T(-L), \Phi_T(L)\rangle \\
&\quad -\lambda^2 \int_{-\infty}^{\infty} \langle \Psi_T(x), B\widehat{\Phi}_T(x)\rangle \, dx + \mathcal{R}_{kj}(\lambda, \gamma)
\end{aligned}$$

$$
\begin{aligned}
&= -4(1-\cos\gamma)q'(L)q''(L) - \lambda^2 \int_{-\infty}^{\infty} q^2 \, dx + \mathcal{R}_{kj}(\lambda,\gamma) \\
&\sim 16(1-\cos\gamma)\mathrm{e}^{-2L} - 2\lambda^2 && (4.25)\\
\xi_{RR} &= (\mathrm{e}^{\mathrm{i}\gamma}-1)\langle \Psi_R(L), \Phi_R(-L)\rangle + (1-\mathrm{e}^{-\mathrm{i}\gamma})\langle \Psi_R(-L), \Phi_R(L)\rangle \\
&\quad -\lambda^2 \int_{-\infty}^{\infty} \langle \Psi_R(x), B\widehat{\Phi}_R(x)\rangle \, dx \\
&= 4(1-\cos\gamma)q(L)q'(L) + 2\lambda^2 \int_{-\infty}^{\infty} q^2 - \frac{1}{2}q^2 \, dx \\
&\sim -16(1-\cos\gamma)\mathrm{e}^{-2L} + 2\lambda^2 && (4.26)
\end{aligned}
$$

Solving $\xi_{RR} = 0$ gives, to leading order,

$$\lambda = \pm\sqrt{8}\sqrt{(1-\cos\gamma)}\,\mathrm{e}^{-L} \tag{4.27}$$

Since $(1-\cos\gamma) \geq 0$, this means that the loops of wave-train spectra arising from the rotation eigenvalue of the pulse lie along the real axis. Thus the dn waves are unstable; this is true whether or not the potential term is present. When $V_0 = 0$, ξ_{TT} is the same as ξ_{RR} to leading order, so in the no-potential case, the loops of wave-train spectra arising from the translation eigenvalue of the pulse also lie along the real axis. Our instability result for dn wave-trains agrees with the observations in [2].

Remark 4.2. Rather than using $q = \operatorname{sech}(x)$, the pulse at $V_0 = 0$, to do the calculations, we can take $q = \sqrt{V_0+1}\operatorname{sech}(x)$. This introduces the factor (V_0+1) into the ξ_{TT}, ξ_{RR} expressions, which gives rise to $\mathcal{O}(V_0\mathrm{e}^{-2L}), \mathcal{O}(\lambda^2 V_0)$ terms. These will appear in the error estimates of the next subsection.

4.4 Error estimates

Estimates for the remainder terms can be obtained as in [5]. In this subsection we summarise the results, and refer to [5] for details. The jump function ξ is analytic in λ, V_0 and can be written as

$$\xi = \xi_0(V_0,L) + \lambda\xi_1(V_0,L) + \lambda^2\xi_2(V_0,L) + \mathcal{O}(|\lambda|^3)$$

The first, $\mathcal{O}(\lambda^0)$, term is obtained by considering $\lambda = 0$, which leads to decoupling of the eigenvalue problem into two independent equations $L_+u = 0, L_-v = 0$. Thus, terms due to translation and terms due to rotation do not interact with each other, and the off-diagonal terms are zero at this level. So

$$\xi_0(V_0,L) = \begin{pmatrix} \xi_{TT}^{(0)} + E_T^{\infty}(V_0) & 0 \\ 0 & \xi_{RR}^{(0)} \end{pmatrix} \tag{4.28}$$

where, extracting the λ-independent terms from subsections 4.2 and 4.3, we have

$$\xi_{TT}^{(0)} = -16(1+\cos\gamma)\mathrm{e}^{-2L} + \mathcal{O}(\mathrm{e}^{-3L} + \mathrm{e}^{-2L}|V_0|) \tag{4.29}$$

$$E_T^{\infty} = 2\cdot\frac{8}{15}V_0 + \mathcal{O}(|V_0|^2) \tag{4.30}$$

$$\xi_{RR}^{(0)} = 16(1+\cos\gamma)\mathrm{e}^{-2L} + \mathcal{O}(\mathrm{e}^{-3L} + \mathrm{e}^{-2L}|V_0|) \tag{4.31}$$

The $\mathcal{O}(\lambda^1)$ term is obtained in a fashion similar to the discussion in subsection 4.1 about modifying the Melnikov integral terms. Differentiating $W' = [A(x) + \lambda B]W$ with respect to λ and setting $\lambda = 0$ gives $w_1' = A(x)w_1 + Bw_0$, where $w_0' = A(x)w_0$. The term $\xi_1(V_0, L)$ comes from $\frac{\partial \xi}{\partial \lambda}\Big|_{\lambda=0} = \langle \Psi(0), w_1^+(0) - w_1^-(0)\rangle$. We have already seen, in subsection 4.1, that the integral $\int_{-\infty}^{\infty} \langle \Psi(x), B\Phi(x)\rangle \, dx$ arising from the inhomogeneous term Bw_0 is zero; however, from the Bloch-wave condition at $\pm L$, inner product terms involving

$$\langle \Psi_T(\pm L), \widehat{\Phi}_R(\mp L)\rangle \quad \text{and} \quad \langle \Psi_R(\pm L), \widehat{\Phi}_T(\mp L)\rangle$$

will appear, which are of order $\mathrm{e}^{-L} \cdot \mathrm{e}^{-L}$. Therefore

$$\xi_1(V_0, L) = \begin{pmatrix} 0 & \mathcal{O}(\mathrm{e}^{-2L}) \\ \mathcal{O}(\mathrm{e}^{-2L}) & 0 \end{pmatrix} \tag{4.32}$$

Another way of seeing the origin of these terms is to observe that $w_1' = A(x)w_1 + Bw_0$ is equivalent to the generalised eigenfunction equations $L_+u_1 = -v_0, L_-v_1 = u_0$, where $u_0 = q'(x)$ and $v_0 = q(x)$. These equations are decoupled from each other; moreover, since L_+ is invertible on even functions and v_0 is even, while L_- is invertible on odd functions and u_0 is odd (see section 2), we can solve each of the two equations for u_1, v_1 on $\mathbb{R}$; thus the only contribution to the jump at $\mathcal{O}(\lambda^1)$ must come from the imposition of boundary conditions at $x = \pm L$.

Finally, the $\mathcal{O}(\lambda^2)$ coeffcients can be extracted from subsections 4.2 and 4.3, namely

$$\begin{aligned} \xi_2(V_0, L) &= \begin{pmatrix} \xi_{TT}^{(2)} & 0 \\ 0 & \xi_{RR}^{(2)} \end{pmatrix} + \mathcal{O}(\mathrm{e}^{-L} + |V_0|) \\ \text{where} \quad & \xi_{TT}^{(2)} = -2, \quad \xi_{RR}^{(2)} = 2. \end{aligned} \tag{4.33}$$

4.5 Main results

Putting together the expressions obtained in subsections 4.2 to 4.4, we have that λ is in the spectrum of the cn wave-train if, and only if,

$$E(\lambda, \gamma) \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{4.34}$$

where

$$\begin{aligned} a_{11} &= -2\left[\lambda^2 - \frac{8}{15}V_0 + 8(1+\cos\gamma)\mathrm{e}^{-2L}\right] \\ &\quad + \mathcal{O}\left[|V_0|(|V_0| + \mathrm{e}^{-2L}) + \mathrm{e}^{-3L} + |\lambda|(|\lambda| + |V_0| + \mathrm{e}^{-L})\right], \end{aligned} \tag{4.35}$$

$$\begin{aligned} a_{22} &= 2\left[\lambda^2 + 8(1+\cos\gamma)\mathrm{e}^{-2L}\right] \\ &\quad + \mathcal{O}\left[|V_0|\mathrm{e}^{-2L} + \mathrm{e}^{-3L} + |\lambda|(|\lambda| + |V_0| + \mathrm{e}^{-L})\right], \end{aligned} \tag{4.36}$$

$$\left.\begin{matrix} a_{12} \\ a_{21} \end{matrix}\right\} = \mathcal{O}\left[|\lambda|\mathrm{e}^{-2L} + |\lambda|^2(|V_0| + \mathrm{e}^{-L})\right], \tag{4.37}$$

and the higher-order remainder terms also satisfy the estimates in γ that are outlined in [5].

This result characterises the position of the spectrum near 0 when k is close to 1 (i.e., $L \gg 1$).

5 Conclusions and discussion

We computed the essential spectrum (there is no point spectrum for spatially periodic waves on $\mathbb{R}$) of the cn waves in the long-wavelength limit for V_0 close to zero and also proved that edge bifurcations occur only for $V_0 > 0$ (and that they do occur in that regime). Our results show spectral instability for all $V_0 > 0$ and spectral stability of the essential spectrum near the origin for $V_0 < 0$ close to zero. Bronski et al. already provided numerical evidence for this in [2] using direct simulations. The global instability result for $V_0 > 0$ is actually based on an "$|N(L_+) - N(L_-)| > 1$" type Grillakis argument for the waves.

We also determined the instability mechanism for $V_0 > 0$: the eigenfunctions which we computed analytically show that the precise instability mechanism depends strongly on the wavelength and the value of V_0. The most unstable mode changes periodically in the wavelength between pairwise attraction/repulsion and a translational instability where all individual pulses begin to move in one direction.

We have not yet determined what happens to the original essential spectrum of the background state when switching to the long-wavelength cn waves (other than looking at edge bifurcation that is). This part of the spectrum ought to stay on the imaginary axis but we have not yet proved that. This should follow from energy considerations (i.e., Krein signatures). We note that computing the essential spectrum of the long-wavelength cn waves is subtle. Sandstede and Scheel [5] developed the relevant theory for the case when $\lambda = 0$ is simple. Here, $\lambda = 0$ has multiplicity four, and all these modes interact for the cn waves even though they are completely decoupled (in the invariant function spaces of even and odd functions) for the solitons. This would be an interesting direction for further study.

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